Closed Loop Control of Purcell's Swimmer

1st Pratik Jawahar
Dept. of Robotics Engineering
Worcester Polytechnic Institute
Worcester (MA), USA

***Abstract*—The Purcell's Swimmer (micro-swimmer) is a really intriguing topic mainly because its dynamics is complicated owing to the Low-Reynolds Number conditions that it is designed to operate in. This makes any intuitive notion of human-like propulsion through a fluid invalid. This project deals with the modeling and simulation of basic controller to implement motion primitives, the three fundamental unit motions for a planar three link swimmer, (pure X-direction, pure Y-direction, pure Θ) which are all defined with respect to the net motion of the center link about its COM.**

***Index Terms*—Purcell's Swimmer, Low Reynolds Number conditions, Geometric swimming, optimal gait, motion primitives**

I. Introduction

Swimming at low Reynold's number is very different from a general notion of swimming primarily because, inertial effects play almost no significant role against the effects of viscosity. This means that any body moving under such conditions will cause the local boundary layer to move along with it, or as stated by E.M.Purcell, "you can't shake off your environment" [2]. A direct implication of this is that any symmetric gait will not produce any net motion per cycle since the distance moved forward in the propulsion stroke will be equal to the distance moved back in the return stroke.

In his paper, Purcell proposes a gait to generate motion for the three link swimmer. This, drift-free three link swimmer is our subject of interest for this paper as shown in Fig. 1

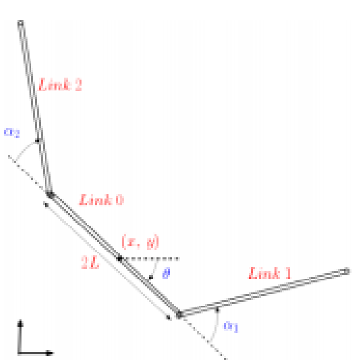

Fig. 1. Purcell's Three Link Swimmer

II. System Models

The Purcell's Swimmer's dynamics is explicitly defined in [1]. We make the assumption of the system existing at very low Reynold's number environments and this implies, the momentum terms are dropped from the *general reconstruction equation* [1]. The general reconstruction equation is given by,

$$\zeta = -A(r)\dot{r} + T(r)p,$$

reducing it to the *kinematic reconstruction equation*,

$$\zeta = -A(r)\dot{r}, \tag{1}$$

where A is the local connection form matrix, ζ is the body velocity vector, $\begin{bmatrix}\dot{x} & \dot{y} & \dot{\theta}\end{bmatrix}^\top$ and $r = \begin{pmatrix}\alpha 1, \alpha 2\end{pmatrix}$, is the joint space parameters defined in Fig.1, with $\dot{r}$ being the joint space velocity vector. The drag forces and moments are thereby found to be functions of ζ. This enables us to write the net force $F = \begin{bmatrix}F_x & F_y & M\end{bmatrix}\top$ as,

$$F = \omega(\alpha)\begin{bmatrix}\zeta \\ \dot{\alpha}\end{bmatrix}$$

which on reducing with considerations for low Reynold's number conditions (net force on isolated system is zero) gives, $A = \omega_1^{-1}\omega_2$ as explained in Sec.[II-C]. We will use this explicit definition of the connection form in later sections of this report.

A. The Parallel Parking Problem

To understand the intuitive meaning behind this system equation better, we look at the parallel parking problem.

The parallel parking problem is quite a fundamental exercise to understand the most basic square gait, a four step drive-steer control sequence and mainly the significance of non-holonomic constraints in day to day life. The Matlab code for this makes our two wheeled car start with the axle centered at the origin, parallel to the X-axis. The car first drives straight up to the first plot in Fig.2. It then performs a steering motion with the axle center being the center of rotation to the position shown in the top right plot of Fig. 2.

The car then drives backward maintaining the angular orientation to the position show in the bottom left plot of Fig. 2. Finally performing a steering maneuver as in Step 2 to orient itself parallel to the X-axis arriving at the position shown in the last plot of Fig.2.

Evidently performing these motions sequentially enables us to have a net translation in a direction that seems impossible to achieve with the given set of available, independent

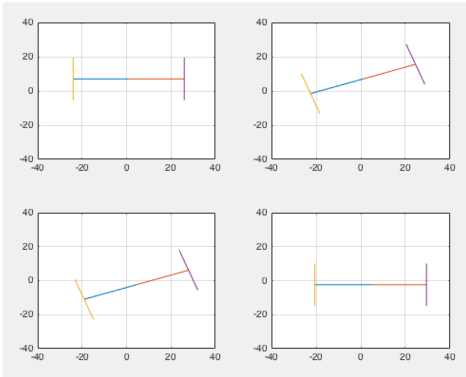

Fig. 2. Plots showing the square-gait, drive-steer control sequence

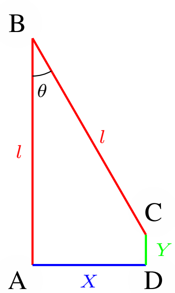

Fig. 3. Diagram to calculate the final position of the car

actuations. There is a negligible deviation in the Y direction, which can be minimized by decreasing the time-step for which each of the symmetric drive and steer gaits are performed.

Fig.3 shows the trajectory of the car (red lines), starting at the bottom left. Taking the length of each red line to be l and the steer angle to be θ, we get the net displacements along the axes as,

$$X = \bar{AD} = \sin\theta$$

$$Y = \bar{CD} = 2l\sin^2\frac{\theta}{2}$$

Here, Y is the error that arises out of the non-holonomy of the system.

B. Purcell's Swimmer System Model

Having understood the intuitive meaning of symmetric gaits, we define our swimmer's system dynamics in order to understand its state space parameters.

The dynamic model described above links the body velocity directly to the shape velocity through the local connection form matrix. First, the connection form matrix $A(r)$ is calculated using the explicit definition given above.

Having found the connection form matrix, we can define the state space equation of our system as,

$$\begin{bmatrix}\dot{\alpha_1}\\ \dot{\alpha_1}\\ \zeta\end{bmatrix} = \begin{bmatrix}1\\ 0\\ A(\alpha_1,\alpha_2)\end{bmatrix}\dot{\alpha_1} + \begin{bmatrix}0\\ 1\\ A(\alpha_1,\alpha_2)\end{bmatrix}\dot{\alpha_2} = g_1(\alpha_1,\alpha_2)\dot{\alpha_1} + g_2(\alpha_1,\alpha_2)\dot{\alpha_2} \tag{2}$$

Here g_1, g_2 are the control vector fields. The control inputs $\dot{\alpha_1}, \dot{\alpha_1}$ are separated so that it is easier to implement an asymmetric gait controller knowing the individual control vector fields g_1, g_2 since it is easier to obtain their Lie Algebra. The flow of the Lie Algebra will in turn work towards identifying gaits that minimize error, thereby forming our primary controller.

C. Calculating Model Parameters

Before we move on to studying the controllability and designing motion primitives, we first have to calculate the Connection Matrix ($'A'$) parameters.

First, we model the forces and moments on the swimmer, under the slender body dynamics assumptions implying the ratio of principal lateral drag coefficient to the longitudinal drag coeffecient being $2:1$ [1],

$$F_{i,x} = \int_{-L}^{L}\frac{1}{2}k\zeta_{i,x}dl = kL\zeta_{i,x}$$

$$F_{i,y} = \int_{-L}^{L}k\zeta_{i,y}dl = 2kL\zeta_{i,x}$$

$$M_i = \int_{-L}^{L}kl(l\zeta_{i,\theta})dl = \frac{2}{3}kL^3\zeta_{i,\theta}$$

where k is the differential viscous drag coefficient.

Next, we calculate the relation between the joint space velocities to the body velocity as,

$$\zeta_1 = \begin{bmatrix} cos(\alpha_1)\zeta_x - sin(\alpha_1)\zeta_y + sin(\alpha_1)L\zeta_\theta \\ sin(\alpha_1)\zeta_x + cos(\alpha_1)\zeta_y - (cos(\alpha_1)+1)L\zeta_\theta + L\dot{\alpha_1} \\ \zeta_\theta\dot{\alpha_1}\end{bmatrix}$$

$$\zeta_2 = \zeta$$

$$\zeta_2 = \begin{bmatrix} cos(\alpha_2)\zeta_x - sin(\alpha_2)\zeta_y + sin(\alpha_2)L\zeta_\theta \\ -sin(\alpha_2)\zeta_x + cos(\alpha_2)\zeta_y + (cos(\alpha_2)+1)L\zeta_\theta + L\dot{\alpha_2} \\ \zeta_\theta\dot{\alpha_2}\end{bmatrix}$$

Next we write the net force and moment equation for all the links of the system as,

$$\begin{bmatrix}F_x\\ F_y\\ M\end{bmatrix} = \begin{bmatrix} cos(\alpha_1)sin(\alpha_1)0 \\ -sin(\alpha_1)cos(\alpha_1)0 \\ Lsin(\alpha_1) - L(1+cos(\alpha_1))1\end{bmatrix}\begin{bmatrix}F_{1,x}\\ F_{1,y}\\ M_1\end{bmatrix} + \begin{bmatrix}F_{2,x}\\ F_{2,y}\\ M_2\end{bmatrix} + \begin{bmatrix} cos(\alpha_2) - sin(\alpha_2)0 \\ -sin(\alpha_2)cos(\alpha_2)0 \\ Lsin(\alpha_2)L(1+cos(\alpha_2))1\end{bmatrix}\begin{bmatrix}F_{3,x}\\ F_{3,y}\\ M_3\end{bmatrix} \tag{3}$$

On combining the above equations, we see that the linear relation between the net force and the velocity vectors is maintained with an additional non-linear dependency on α. Thus, we can write the above equation as,

$$\begin{bmatrix} F_x \\ F_y \\ M \end{bmatrix} = \omega(\alpha)^{3\times 5} \begin{bmatrix} \zeta \\ \dot{\alpha} \end{bmatrix} \quad (4)$$

We split $\omega(\alpha)$ as 3×3 appended with a 3×2 in order to split the velocity terms and get an explicit relation between the two. [3]

$$\begin{bmatrix} F_x \\ F_y \\ M \end{bmatrix} = \begin{bmatrix} \omega_1^{3\times 3} \omega_2^{3\times 2} \end{bmatrix} \begin{bmatrix} \zeta \\ \dot{\alpha} \end{bmatrix} \quad (5)$$

Since the system is at Low Reynolds number conditions, the major implication on the system is that the net external forces and moments on the isolated system are zero. Thus,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_1^{3\times 3} \omega_2^{3\times 2} \end{bmatrix} \begin{bmatrix} \zeta \\ \dot{\alpha} \end{bmatrix} \quad (6)$$

This gives us a relation between the body velocity vector and the joint space velocity vector as,

$$\zeta = -\omega_1^{-1} \omega_2 \dot{\alpha} \quad (7)$$

Comparing Eqn. (7) and (1), we can calculate the local connection matrix A and thus the model definition is complete.

III. Controllability

Since our system is non-linear and encloses non-holonomic constraints, instead of defining stability and controllability in terms of the Lyapunov definition, we consider the smooth integral manifolds the system can traverse and define controllability in terms of the Lie Algebra of the control vector fields using Chow's Theorem. [4]

In this regard, we first define strong and weak controllability according Chow's definition based on the traversable set of smooth manifolds.

A system is said to be strongly controllable if for any given initial joint state (q_0) and final joint state (q_f), there exists a time $T > 0$ such that the system state at $t = 0$ is q_0 and at $t = T$ is q_f. Also, there exists a curve $X(t)$ passing through q_0 having $X(0) = q_0$ and $X(T) = q_f$.

A system is said to be weakly controllable if for any $q_0 = [x_0 g_0]$ and $q_f = [x_f g_f]$, where x, g are the position and shape respectively, if there exists a time $T > 0$ and a base curve $X(t)$ such that it satisfies $X(0) = q_0$ and the horizontal lift of the base curve satisfies $X(T) = x_f$.

With these definitions in place we plan our controller to use the manifold traversal equation to reduce the error.

IV. Control Law

The non-holonomic constraints on our system make achieving strong controllability at all points a highly complex task. So we design a control law using the weak definition for controllability, and this control law will serve as our position controller for the three motion primitives, pure-X, pure-Y and pure-θ.

According to Chow's theorem, the system is said to be weakly controllable if,

$$span(LieAlgebrae(g_1, g_2)) = span(T_q(Q))$$

where $T_q(Q)$ are the traversable states along a given smooth manifold. For a drift-less system, $span(T_q(Q)) = 5$ [3]. Thus, to ensure controllability, we have to ensure $span(LHS) = 5$. This is done by neglecting the higher order Lie Algebra terms.

$$span(g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]]) = span(T_q(Q) \quad (8)$$

So, to ensure weak controllability, our control law should be a linear combination of the Lie Algebra terms is Eqn.8. We calculate the co-efficients of each of the terms as follows,

$$\begin{bmatrix} g1 & g2 & [g_1, g_2] & [g_1, [g_1, g_2]] & [g_2, [g_1, g_2]] \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \\ C4 \\ C5 \end{bmatrix} = \begin{bmatrix} B1 \\ B2 \\ B3 \\ B4 \\ B5 \end{bmatrix} \quad (9)$$

where, $C_i(1i5)$ is the matrix of coefficients and $B_i(1i5)$ is the desired reference state (orientation-position) matrix.

Expanding each Lie Algebra solving and solving Eqn.9, we get,$C_1 = C_2 = 0$ and the other three are say, α, β, γ. These are calculated using a Force analysis of the model parameters as explained in [3]. Once we have the co-efficients matrix, we define the control law as the net flow composition equation [3] given below which acts as the desired motion controller.

$$
\begin{aligned}
&\phi_t^{\alpha[g_1,g_2]\beta[g_1,[g_1,g_2]]\gamma[g_2,[g_1,g_2]]} \\
&= \lim_{n\to\infty}[(\phi_{t/n}^{\alpha[g_1,g_2]} \circ \phi_{t/n}^{\beta[g_1,[g_1,g_2]]}\circ \\
&\quad \phi_{t/n}^{\gamma[g_2,[g_1,g_2]]})^n] \\
&= \lim_{n\to\infty}[(\phi^{-g_2}_{\sqrt{t}/n}\circ\phi^{-\alpha g_1}_{\sqrt{t}/n}\circ\phi^{g_2}_{\sqrt{t}/n}\circ \\
&\quad \phi^{\alpha g_1}_{\sqrt{t}/n})^n\circ \\
&(((\phi^{g_2}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{g_1}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{-g_2}_{\sqrt{(\sqrt{t})}/n}\circ \\
&\quad \phi^{-g_1}_{\sqrt{(\sqrt{t})}/n})^n)\circ\phi^{-\beta g_1}_{\sqrt{t}/n}\circ \\
&((\phi^{-g_2}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{-g_1}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{g_2}_{\sqrt{(\sqrt{t})}/n}\circ \\
&\quad \phi^{g_1}_{\sqrt{(\sqrt{t})}/n})^n)\circ\phi^{\beta g_1}_{\sqrt{t}/n})^n] \\
&(((\phi^{g_2}_{\sqrt{(\sqrt{t})}/n}\circ \\
&\phi^{g_1}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{-g_2}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{-g_1}_{\sqrt{(\sqrt{t})}/n})^n)\circ\phi^{-\gamma g_2}_{\sqrt{t}/n}\circ \\
&((\phi^{-g_2}_{\sqrt{(\sqrt{t})}/n}\circ \\
&\phi^{-g_1}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{g_2}_{\sqrt{(\sqrt{t})}/n}\circ\phi^{g_1}_{\sqrt{(\sqrt{t})}/n})^n)\circ\phi^{\gamma g_2}_{\sqrt{t}/n})^n]^n
\end{aligned}
$$

This complex equation has a very simple intuitive meaning. It essentially is a linear combination of the three motion primitive generators defined by the terms corresponding to C_3, C_4, C_5. These terms are calculated from the desired reference state matrix and correspond to the position and orientation of the base link. Since the system is found to be weakly controllable, we can only control system to reach a final desired position and orientation by traversing the corresponding smooth manifold. The final shape parameters of the system cannot be controlled in its reachable space.

At each time step, new co-efficients are calculated from the position and orientation of the next time step, and are fed to the flow equation. The flow equation crunches out the required gait since g_1, g_2 correspond to the joint state parameters α_1, α_2. This can essentially though of as an on-off sequential controller.

The terms that become zero represent 'off gaits' and the rest are 'on gaits' and what terms are turned on or off are defined by the co-efficients calculated at each time-step. These inturn correspond to the state error of the base link, and thereby the system is pushed to converge towards the reference position.

However, the system can only reach the reference point by travelling along the traversible manifolds, meaning trajectory tracking becomes a very arduous task.

V. Simulation Results

The code flow for the simulation is given as,

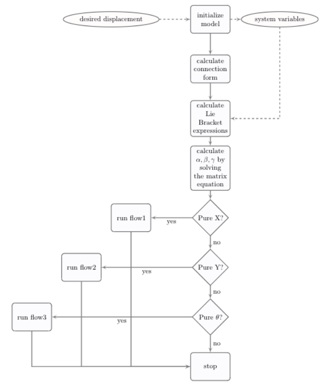

Fig. 4. Simulation Code Flow

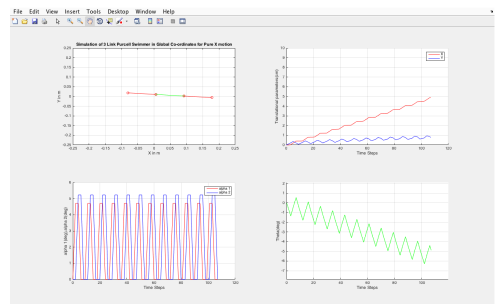

Fig. 5. Pure-X Motion, Gait Simulation

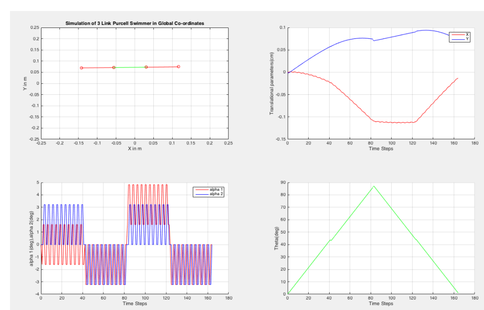

Fig. 6. Pure-Y Motion, Gait Simulation

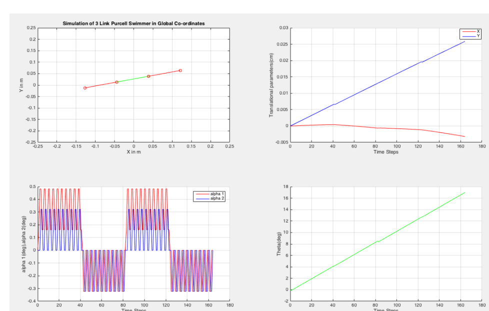

Fig. 7. Pure-θ Motion, Gait Simulation

The simulation results are shown in Figs. 5,6,7.

From the figures, we see how each parameter changes for pure-X, pure-Y and pure-θ motion respectively. The plot on the bottom left of each image represents the gait calculated by the flow control law. These gaits are essentially the α and $\dot{\alpha}$ values obtained from the flow equation, and are executed with a time lag of $\sqrt{\frac{t}{n}}$ in the order prescribed by the flow equation. This is a lucid example of how assymetric the gaits need to be to achieve motion at Low Reynolds numbers.

It is evident from the plots of the x, y, θ parameters, that the control law behaves as an on off controller with the error value in the non-objective parameters averaged over each gait cycle. This is the reason for the triangular wave-like characteristic of the non-objective parameters in each plot.

VI. Conclusions Future Work

The motion primitives, pure-X, pure-Y and pure-θ were successfully achieved with acceptable error in other two state parameters for each case as shown in Sec. V. The future scope for work involves designing a controller that is robust to error correction by chosing the optimal correction strategy and selecting optimal gaits that will thereby enable efficient trajectory tracking despite the existence of non-holonomic constraints. The steps that need to be realised along with this, for accurate real world simulation would be to model and design a separate controller on top of the trajectory tracking controller to model and negate the natural drift in the system.

References

[1] Ross L. Hatton and Howie Choset, *Geometric Swimming at Low and High Reynolds Numbers*, IEEE Transactions on Robotics, 29(3):615–624, 2013.

[2] Edward M Purcell, *Life at Low Reynolds Numbers*,American journal of physics, 45(1):3–11, 1977.

[3] Kedar Joshi and Naman Gupta and Pulkit Katdare and Sudin Kadam and Ravi N. Banavar, *Trajectory Tracking Using Motion Primitives for the Purcell's Swimmer*, CoRR, abs/1703.06731, 2017.

[4] Sanjay P. Bhatt, *Controllability of Non-Linear Systems*, Lecture Notes.